International Journal of Multidisciplinary and Current Educational Research (IJMCER)
ISSN: 2581-7027 ||Volume|| 1 ||Issue|| 3 ||Pages|| 01-05 ||2019||

An Effective Teaching Technical to Build a Homographic Recurring Sequence

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Summary: *Their age, their richness as well as the diversity of their fields of application make recurring series a so vast and so rich subject in results that it would take several works, in addition to those that already exist, to go around all their properties. In the literature, we mainly learn about linear recurrent sequences. But, this paper is interested in studding homographic recurring sequence. Apart from a reminder about the study of the convergence of a recurring homographic sequence, we set up two theorems allowing to build a homographic sequence whose limit is beforehand fixed.*

KEY WORDS: Real digital suite, homographic function, geometric suite, continued arithmetic.

I. INTRODUCTION

In mathematics, a sequence is an ordered succession of elements taken in a given set; a series is the sum of the terms of a sequence (Nicolas, M. 2007). Sequence and series occupy a fundamental place in modern mathematics. The work of Abel, Cauchy and Gauss on convergence marked, in the early nineteenth century, the study of series. This one is not limited to series of real numbers, but also applies to complex numbers, or series of functions (Abdelkader, N. 1998). Series have applications in many scientific fields, like electronics (Hoggat, V. E. 1983), (Cerruti, U., and Vaccarino, F. 1996). There are several types of suites in the literature, namely for example following recurring, Cauchy sequence, Fibonacci sequence, geometric sequence, arithmetic sequence, etc. and each of these sequence has its reasons to be (Roland, C. 2005), (Ferrand, D. 1988), (Hansel, G. 1986), (S. Homer and J. Goldman, 1985). In this paper, we saoudite importance of both geometric and arithmetic sequence. Nobody is ignorant of the importance of geometric sequence and arithmetic sequence (Pourchet, Y. 1979), (Polosuev, A. M. 1986) in everyday life, and this, in several domains, namely, for example: bank, administration, medicine, bio-ecology, telecommunication, etc. In this regard, this paper proposes, among other things, a way allowing to obtain recurring sequence from homographic functions (Michard, R. 2008) et (Mikhalev et al. 1995). In what follows, our work is divided into four sections. Section II introduces the definition of an arithmetic sequence and that of geometric. Section III suggests the direct problems of the homographic. Section IV proposes the inverse problems of homography. Section V poses a conclusion.

II- REMINDERS AND DEFINITIONS

In this section, we recall a little the definition of an arithmetic and geometric sequence.

2.1. Arithmetic sequence

An arithmetic sequence is called a sequence of numbers where we go from one term to the next by adding always the same number (this number is called the reason for the arithmetic sequence and is often noted *r*) (A. M. Polosuev. 1967), (J.-P. Bézivin. 1990), (M. Mignotte L. Cerlienco and F. 1987).

2.2. Geometric sequence

A geometric sequence is a numerical sequence whose term is obtained by multiplying previous by a nonzero constant real number *q* (it is a recursion definition).

III- STUDY OF A RECURRENT SEQUENCE " DIRECT PROBLEMS OF HOMOGRAPHY "

3.1. Linear Recurrence Suite: Classical Problem (on the Student Side)

Let $(U_n)_{n\in IN}$ be the sequence defined by $U_0 = 1$ and for all *n*: $U_{n+1} = 3U_n + 5$.

- ✓ For all n, we set $V_n = U_n - \frac{5}{2}$

a- Show that the sequence (V_n) is a geometric sequence whose exact reason and his first term.
b- Express $(V_n)_{n\in IN}$ then $(U_n)_{n\in IN}$ according to n
c- Calculate the limit of the sequence $(V_n)_{n\in IN}$ and deduce that $(U_n)_{n\in IN}$

3.2. Homographic recurring sequence proper : classical problem (on the side of students)

We want to study the convergent of a sequence $(U_n)_{n\in IN}$ expressed by the relation of recurrence $U_{n+1} = h(U_n)$ from the situations of the fixed points of an own homography h previously given.
Let's start by exposing classical methods for the study of some recurrent sequences of order one. If a $(U_n)_{n\in IN}$ sequence is defined by recurrence by an expression of the type $U_{n+1} = h(U_n)$, where h is a homographic function defined by: $h(x) = \frac{ax+b}{cx+d}$ (c≠ 0 otherwise the study is trivial), then of two things one:

- ✓ Let the function h admit two fixed points α and β in which case we study the term sequence general $V_n = \frac{U_n-\alpha}{U_n-\beta}$ and we quickly realize that $(V_n)_{n\in IN}$ is a geometric sequence and we draw from it after the convergence of the sequence $(U_n)_{n\in IN}$.
- ✓ Else, h has only one fixed point, in which case we study the following general term $V_n = \frac{1}{U_n-\alpha}$ which is then an arithmetic sequence; in this case, we can shoot immediately that $\lim_{n\to\infty} U_n = \alpha$.

3.3. Development of an auxiliary geometric sequence

If a sequence $(U_n)_{n\in IN}$ is defined by recurrence by an expression of the type : $U_{n+1} = h(U_n)$ with h a homography of the form $x \mapsto \frac{ax+b}{cx+d}$ (c≠ 0 otherwise the study is trivial), then of both things one : either the function h has two fixed points and in which case we study the sequence of general term $V_n = \frac{U_n-\alpha}{U_n-\beta}$, and we quickly realize that (V_n) is a geometric sequence of reason $q = \frac{a-c\alpha}{a-c\beta}$ and of first term V_p such that $V_p = \frac{U_p-\alpha}{U_p-\beta}$ hence, the following proposition.

Proposition 1. *Given homography h having two fixed points and considering a homographic sequence* $U_{n+1} = h(U_n)$, *then we have the 4 possible cases following the values of the reason of the auxiliary sequence* (V_n) *:*
(i) *If* $|q|> 1$, *then the sequence* $(U_n)_{n\in IN}$ *converge to* α *;*
(ii) *If* $|q|< 1$, *then the sequence* $(U_n)_{n\in IN}$ *converge to* β *;*
(iii) *If* $|q| = 1$, *and the first term* $V_p \neq 1$, *then the sequence* $(U_n)_{n\in IN}$ *converge to* $\pm\frac{\alpha-\beta V_p}{1-V_p}$ *;*
(iv) *Else, then the sequence* $(U_n)_{n\in IN}$ *diverge.*

Proof
(i) Let $(U_n)_{n\in IN}$ be a sequence defined by : $U_{n+1} = h(U_n)$, such that there are two non-zero reals and verifying the equations h(α)= α and h(β) =β. Consider a suite $V_n = \frac{U_n-\alpha}{U_n-\beta}$. As, we have already said that $(V_n)_{n\in IN}$ is a geometric sequence of reason $q = \frac{a-c\alpha}{a-c\beta}$ and of first term V_p such that $V_p = \frac{U_p-\alpha}{U_p-\beta}$. We can deduce from this that the general term of the $(V_n)_{n\in IN}$ sequence of first term Vp and of reason q is defined by: $V_n = \frac{U_n-\alpha}{U_n-\beta} = q^{n-p}V_p = \left(\frac{a-c\alpha}{a-c\beta}\right)^{n-p}\frac{U_p-\alpha}{U_p-\beta}$
We can indeed draw that, if |q| > 1, then:

$$\lim_{n\to\infty}\frac{U_n-\alpha}{U_n-\beta} = \lim_{n\to\infty}\left(\frac{a-c\alpha}{a-c\beta}\right)^{n-p}\frac{U_p-\alpha}{U_p-\beta} = \infty \qquad (1)$$

Which leads us to deduce that $U_n = \beta$. It is indeed very easy to prove that, *(ii)*, *(iii)* and *(iv)* results from (1).
Example 1. *Let us study the sequence defined by* $U_0 = 1$ *and for all n:* $U_{n+1} = \frac{U_n+3}{2U_n}$. *We start by looking for the fixed points of homography h:* $x \mapsto \frac{x+3}{2x}$. *These are the roots of the equation x + 3 = x (2x), that is -1 and 3/2. We are therefore studying the following general term* $\frac{U_n+1}{U_n-3/2}$ *and let's call it* (V_n).
We have:

$$V_{n+1}=\frac{U_{n+1}+1}{U_{n+1}-3/2}=\frac{\frac{U_n+3}{2U_n}+1}{\frac{U_n+3}{2U_n}-3/2}$$

$$=\left(-\frac{3}{2}\right)\frac{U_n+1}{U_n-3/2}$$

$$=\left(-\frac{3}{2}\right)V_n.$$

So, $(V_n)_{n\in IN}$ *is a geometric sequence of reason* $q=-\frac{3}{2}$ *and of first term* $V_0=4$.

3.4. Convergence of a suite $(U_n)_{n\in IN}$

By recurrence, we have immediately $\frac{U_n+1}{U_n-3/2}=(-3/2)^n\frac{U_0+1}{U_0-3/2}$ and therefore $\lim_{n\to\infty}\left|\frac{U_n+1}{U_n-3/2}\right|=+\infty$. Hence, $\lim_{n\to+\infty}U_n=3/2$.

3.5. Development of an auxiliary arithmetic suite

As we have already shown that if *h* has only one fixed point, then the following general term $V_n=\frac{1}{U_n-\alpha}$ is then an arithmetic sequence of reason $r=\frac{c}{a-\alpha c}$ and first term $V_p=\frac{1}{U_p-\alpha}$. In this case, we can immediately draw that $\lim_{n\to+\infty}U_n=\alpha$.

Example 2. *We will study the sequence defined by* $U_0=1$ *and for all n:* $U_{n+1}=\frac{10U_n-25}{U_n}$. *We start by searching for the fixed points of the recursion function f:* $x\mapsto\frac{10x-25}{x}$. *These are the roots of the equation 10x - 25 = x (x), that is to say 5. We therefore study the following auxiliary general term* $\frac{1}{U_n-5}$ *let's call it* $(V_n)_{n\in IN}$.

We have:

$$V_{n+1}=\frac{1}{U_{n+1}-5}=\frac{1}{\frac{10U_n-25}{2U_n}-5}$$

$$=\left(\frac{1}{5}\right)\frac{U_n}{U_n-5}$$

So, $V_{n+1}-V_n=\frac{1}{5}$.

Hence, $(V_n)_{n\in IN}$ *is an arithmetic sequence of reason* $r=1/5$ *and first-term* $V_0=-\frac{1}{4}$.

3.6. Convergence of the suite $(U_n)_{n\in IN}$

It is immediately that $\lim_{n\to+\infty}U_n=5$. This method is certainly effective and it is put through the search of the fixed points of the homographic function which makes it possible to determine where the sequence defined by recurrence may tend, if it converges. She is too actually place expected to have a projective application that can make the infinite to a finite number. On the other hand, it is more surprising that this method gives us for sure the behavior of the suite.

IV- INVERSE PROBLEMS

4.1. Inverse problems of an affine application

On the teacher's side, "how to construct one's own subject on linear sequences or arithmetico-geometric?" It's about finding an affine application with a fixed point given. What will be used to build an arithmetic-geometric sequence that is used to teach to students of terminal or first as part of the study of geometric sequence. Hence the following theorem.

Theorem 1. *Search for an infinity of an affine application having α fixed point given for any non-zero real, there is an infinity of affine applications noted* h_α *having α as fixed point defined by:* $h_\alpha(z)=az+b$ *were* $a\neq 0$ *as,* $=a(1-\alpha)\ \ \forall a\in\mathbb{R}^*\ with\ |a|<1$.

Corollary 1. *For every real fixed α, there is an infinity of arithmetic-geometric sequences that converge to α. We have now a method to build an arithmetico-geometric sequences converging toward α with one degree of freedom.*

Example 3. *Construction of a homography with a fixed point α. Let two real α = 3 and a = 1/4. The arithmetic-geometric sequence generated by h_α is defined by:* $\begin{cases} U_0 \neq 3 \\ U_{n+1} = h_\alpha(U_n) \end{cases}$ *where,* $h_{\alpha z} = \frac{1}{4}z + \frac{9}{4}$ *as* $b = a(1-\alpha) = \frac{9}{4}$ *is*

$$\begin{cases} U_0 \neq 3 \\ \forall n \in \mathbb{N}, U_{n+1} = h_\alpha(U_n) = \dfrac{1}{4}U_n + \dfrac{9}{4}. \end{cases}$$

It is easy to verify that this arithmetic-geometric sequence converges to 3 as desired.

4.2. Inverse problems of proper homography

This time, we propose to build a boundary homographic sequence fixed at will. This is a reverse problem of homography which consists in finding:

- Let a complex or real homography *h*, which α is precisely its only fixed point ;
- Let a complex or real *h* homography at two fixed points of which α.

Let's examine the two cases separately.

4.3. A complex or real homography *h* of which is its only fixed point

Theorem 2. *For α fixed non-zero real, there is an infinity of homographs marked h_α having as α single fixed point defined by:*

$$h_\alpha : z \mapsto \frac{az+b}{cz+d} \text{ as } \begin{cases} b = -\alpha^2 c \\ a = 4\alpha c \\ d = 2\alpha c \end{cases} \quad \forall (\alpha, c) \in (\mathbb{R}^*)^2.$$

Proof

Let h_α be a homography defined by: $h_\alpha(z) = \frac{az+b}{cz+d}$. We have an expression: $h_{\alpha c}(z) = \frac{4\alpha z - \alpha^2}{z+2\alpha}$. Thus, $h_\alpha(z) = z$ is equivalent to $4\alpha cz - \alpha^2 c = cz^2 + 2\alpha cz$. So, in replacing z by α we have: $4\alpha^2 c - \alpha^2 c = c\alpha^2 + 2\alpha^2 c$ and we have: $3\alpha^2 c = 3c\alpha^2$.

Uniqueness

We have: $z^2 + 2\alpha z + \alpha^2 = (z+\alpha)^2$. We find that the discriminant of this equation is $\Delta = 0$. Hence, the theorem stated.

Note there are thus two degrees of freedom on the choices of α and c to build a homographic sequence converging toward α and such that the homographic function of recurrence admits as α single fixed point as desired.

Corollary 2. *There are infinitely many homographic $(U_n)_{n\in\mathbb{N}}$ sequences that converge to α. We have now a method to build on a convergent homographic suite toward α two degrees of freedom.*

Example 4. *Let two reals α = 1 and c = 2. The homographic sequence generated by $h_{\alpha c}$ is defined b :*

$$\begin{cases} U_0 \neq 1 \\ U_{n+1} = h_{\alpha c}(U_n) \end{cases} \quad as \quad \begin{cases} \alpha = 1 \\ c = 2 \\ b = -2 \\ a = 8 \\ d = 4. \end{cases}$$

So, $\begin{cases} U_0 \neq 1 \\ \forall n \in \mathbb{N}, U_{n+1} = h_{\alpha c}(U_n) = \frac{4U_n - 1}{U_n + 2}. \end{cases}$

It is easy to verify that this homographic sequence converges to 1 as desired.

4.4. A complex or real homography *h* two fixed points

Theorem 3. *For two real α and β fixed, there is an infinity of noted homographies, $h_{\alpha\beta}$, having and as the only fixed point defined by :*

$$h_{\alpha\beta} : z \mapsto \frac{az+b}{cz+d} \text{ as } \begin{cases} b = -(\alpha\beta)c \\ a = 2(\alpha+\beta)c \\ d = (\alpha+\beta)c \end{cases} \quad \forall (\alpha, c, \beta) \in (\mathbb{R}^*)^3.$$

Proof

Let $h_{\alpha\beta}$ be a homography defined by: $h_{\alpha\beta}(z) = \frac{az+b}{cz+d}$. We have an expression: $h_{\alpha\beta c}(z) = \frac{2(\alpha+\beta)-(\alpha\beta)}{z+(\alpha+\beta)}$. Thus, $h_{\alpha\beta}(z) = z$ is equivalent to $2(\alpha+\beta)cz - (\alpha\beta)c = cz^2 + (\alpha+\beta)cz$. So, in replacing z by α we have: $2(\alpha +$

$\beta)c\alpha - (\alpha\beta)c = c\alpha^2 + (\alpha+\beta)c\alpha$ and we have: $2\alpha^2 c + \alpha c\beta = 2\alpha^2 c + \alpha c\beta$ and in replacing z by β we have $2\alpha^2 c + \alpha c\beta = 2\alpha^2 c + \alpha c\beta$. So, α and β are two solutions of the equation $h_{\alpha\beta}(z) = z$. Hence, the is stated theorem.

Corollary 3. *There are infinitely many homographic $(U_n)_{n\in\mathbb{N}}$ sequences such as homographies corresponding to two fixed points previously given α and β. As for the first case above, we have a safe technique to build a homographic sequence with two fixed points α and β this time having three degrees of freedom with the choice of α, β and c.*

Example 5. Let three reals α = -1, $\beta = 2$ and c = -3. The homographic sequence generated by

$h_{\alpha c}$ is defined by : $\begin{cases} U_0 \neq \{-1\,;-2\} \\ U_{n+1} = h_{\alpha c}(U_n) \end{cases}$ *as* $\begin{cases} \alpha & = -1 \\ \beta & = 2 \\ c & = -3 \\ b & = -6 \\ a & = -6 \\ d & = -3. \end{cases}$

So, $\begin{cases} U_0 \neq \{-1\,;-2\} \\ \forall n \in \mathbb{N}, U_{n+1} = h_{\alpha c}(U_n) = \frac{2U_n+2}{U_n+1} \end{cases}$ *is a homographic sequence whose corresponding homography has two fixed points -1 and 2.*

V. CONCLUSION AND PERSPECTIVES

In a word, it is possible to construct various problems based on a homographic sequence at will depending on whether one can control or reinforce the acquisition of arithmetic sequences or geometrics with regard to the properties of the fixed points of a homography to study geometric and arithmetic sequences in 11th Grade and in 12th Grade. In terms of learning homographic functions and homographic recurrent sequences in the scientific terminal, we have seen that the school curriculum does not feel the slightest consideration, said homography. Moreover, thanks to the resolution of the inverse problems of homography, we have shown the existence of an infinity of homography at a single given fixed point or at two given fixed points. This allows any teacher to effectively develop his own limit homographic sequences previously fixed at will, and this, in an infinity of ways by having two or three degrees of freedom. Finally, would it not be appropriate to launch the introduction of these methods through studying of convergence of a recurring homographic sequence that is very realizable on mathematics schooling to introduce at the high school level, as reinforcement?

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